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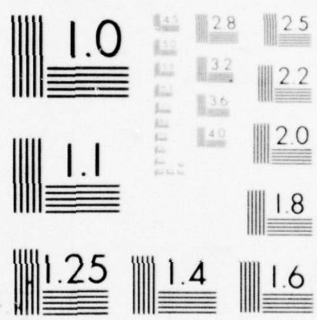

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Let $\gamma_1, \ldots, \gamma_n$ be complex constants. The set $W_{(\gamma_1,\ldots,\gamma_n)}(A) = \{\sum \gamma_u (Ax_j, x_j)\}$, where $(x_1,\ldots,x_n)$ vary over all orthonormal systems in C^n, is called a generalized numerical range of a given $n \times n$ matrix A. In this paper we study inclusion relations of the form $W_{(\gamma_1,\ldots,\gamma_n)} \subset$

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$\lambda W_{(\gamma_1',\ldots,\gamma_n')}$ which hold uniformly for all n-square matrices A. In particular we concentrate on the case where the coefficients are real. Such inclusion relations yield simple inequalities among <u>generalized numerical radii</u>. Finally, a further generalization of the above numerical range is discussed.

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Elementary Inclusion Relations for Generalized Numerical Ranges

by

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Dedicated to Olga Taussky Todd

ABSTRACT. Let $\gamma_1,\cdots,\gamma_n$ be complex constants. The set $W_{(\gamma_1,\cdots,\gamma_n)}(A) = \left\{\Sigma\, \gamma_j(A x_j, x_j)\right\}$, where $(x_1,\cdots,x_n)$ vary over all orthogonal systems in $\mathbf{C}^n$, is called a generalized numerical range of a given $n \times n$ matrix A. In this paper we study inclusion relations of the form $W_{(\gamma_1,\cdots,\gamma_n)} \subset \lambda W_{(\gamma_1',\cdots,\gamma_n')}$ which hold uniformly for all n-square matrices A. In particular we concentrate on the case where the coefficients are real. Such inclusion relations yield simple inequalities among generalized numerical radii. Finally, a further generalization of the above numerical range is discussed.

1. Introduction

Let A be an $n \times n$ complex matrix; let $c = (\gamma_1,\cdots,\gamma_n) \in \mathbf{C}^n$ be a fixed complex vector, and let Λ_n be the set of all orthonormal n-tuples of vectors in $\mathbf{C}^n$. In this paper we study some inclusion relations between generalized numerical ranges which are sets in the complex plane of the form

$$W_c(A) = W_{(\gamma_1,\cdots,\gamma_n)}(A) = \left\{ \sum_{j=1}^{n} \gamma_j (A x_j, x_j) : (x_1,\cdots,x_n) \in \Lambda_n \right\}.$$

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From the definition it is clear that $W_c(A)$ actually depends only on the unordered set $\{\gamma_1,\cdots,\gamma_n\}$ rather than on the ordered n-tuple $c = (\gamma_1,\cdots,\gamma_n)$. In the following the vector c will always stand as a representative of the set $\{\gamma_1,\cdots,\gamma_n\}$ and we write $c \sim c'$ if c and c' represent the same set.

We recall now the definition of the k-numerical range given by Halmos [1, §167], which after a simple normalization becomes,

$$W_k(A) = \left\{\frac{1}{k}\operatorname{tr}(PAP) : P = \text{projection of rank } k\right\}, \quad (1 \leq k \leq n) .$$

Evidently $W_k(A)$ may be written as

$$W_k(A) = \left\{\frac{1}{k}\sum_{j=1}^{k} (A x_j, x_j) : (x_1,\cdots,x_k) \in \Lambda_k\right\} \tag{1.1a}$$

where Λ_k is the set of all k-tuples of orthonormal vectors in $\mathbb{C}^n$. Hence we see that

$$W_k(A) = W_c(A) \quad \text{with} \quad c = \frac{1}{k}(e_1 + \cdots + e_k) , \tag{1.1b}$$

$\{e_j\}_{j=1}^n$ being the standard basis for $\mathbb{C}^n$. Thus, the k-numerical range is a special case of a generalized numerical range. In particular, for $k = 1$, i.e., for $c = e_1$, we obtain the classical range

$$W(A) = W_1(A) = \{(A x, x) : |x| = 1\} .$$

It is also clear that

$$W_n(A) = \left\{\frac{1}{n}\operatorname{tr} A\right\} .$$

Berger, [1, §16.7], has shown that $W_k(A)$ is convex. It was later proven by Westwick, [2], that $W_c(A)$ is convex for any $c \in \mathbb{R}^n$. Westwick also gave an example which shows that for complex vectors $c \in \mathbb{C}^n$ with

$n \geq 3$, the range $W_c(A)$ may fail to be convex.

Certain inclusion relations involving k-numerical ranges were given in [3]. As in [3], we are interested here in inclusions which hold uniformly for all $A \in C_{n\times n}$, that is for all $n \times n$ complex matrices. In this paper we shall restrict our attention to elementary inclusion relations, i.e, relation of the simple form

$$(1.2) \qquad W_c(A) \subset \lambda W_{c'}(A), \quad \lambda = \text{constant}.$$

In a forthcoming paper we shall consider inclusion relations involving finite linear combinations and integrals of generalized numerical ranges.

We begin in Section 2 with some definitions. This leads, in Section 3, to the construction of inclusion relations of type (1.2) for the general case $c, c' \in C^n$. Further results are obtained in Section 4 for the case $c, c' \in \mathbb{R}^n$. In Section 5, we derive some inequalities among generalized numerical radii. Finally, in Section 6, we define a further, and in a certain sense an ultimate generalization of the concept of numerical range.

2. Partial order relations

We begin by defining two partial order relations among complex vectors.

DEFINITION 1. (i) For $c = (\gamma_1, \cdots, \gamma_n)$ and $c' = (\gamma'_1, \cdots, \gamma'_n)$ in C^n we say that $c \prec c'$ if there exists a doubly stochastic matrix S (i.e., a matrix with non-negative entries whose row and column sums are 1), such that $c = Sc'$.

(ii) The vector c is obtained from c' by <u>pinching</u> if two components γ'_i, γ'_j of c' are replaced by γ_i, γ_j with

$$(2.1) \qquad \gamma_i = \alpha\gamma_i' + (1-\alpha)\gamma_j', \quad \gamma_j = (1-\alpha)\gamma_i' + \alpha\,\gamma_j' \; ; \quad 0 \le \alpha \le 1 ,$$

while the other components of c remain unchanged. Note that pinching an n-tuple c' consists of moving two of its components towards their midpoint, and thus decreasing

$$\mathrm{conv}(c') \equiv \text{convex hull } \{\gamma_1', \cdots, \gamma_n'\} .$$

A similar concept of pinching was used in [4] by Horn and Steinberg.

(iii) We say that $c \ll c'$ if c is obtained from c' by a succession of a finite number of pinchings.

Note that the relations $\prec, \ll$ are in fact relations between the unordered n-tuples $\{\gamma_1, \cdots, \gamma_n\}$ and $\{\gamma_1', \cdots, \gamma_n'\}$. In case (i) it follows from the fact that doubly stochastic matrices are closed under multiplications by permutation matrices. For case (iii) it follows directly from the definition.

THEOREM 1. *The relation* $c \ll c'$ *implies* $c \prec c'$ *but not conversely*.

Proof. If $c \ll c'$, then assume for simplicity that c has been obtained from c' by a single pinch. Hence, for some $i,j \in \{1,\cdots,n\}$ and α with $0 \le \alpha \le 1$, we have (2.1). So $c = Sc'$, where S is the doubly stochastic matrix defined by

$$S_{pq} = \begin{cases} 1 & p = q \ne i,j , \\ \alpha & (p,q) = (i,i),(j,j) , \\ 1-\alpha & (p,q) = (i,j),(j,i) , \\ 0 & \text{otherwise} . \end{cases}$$

Consequently $c \prec c'$ and the first part of the proof is established.

Next consider the vectors $c = (1/2, i/2, 1/2 + i/2)$ and $c' = (0,1,i)$. Clearly

$$(2.2) \qquad c = Sc' \quad \text{with} \quad S = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix},$$

so $c \prec c'$. However the components of c are all located on the different edges of $\operatorname{conv}(c)$. Therefore, any chain of non-trivial pinches on c' yields a vector c'', where at least two components of c are outside $\operatorname{conv}(c'')$. Hence $c \neq c''$ and the relation $c \ll c'$ fails to hold.

We now wish to show that $\prec$ is a partial order relation. For this purpose we need the next lemma which seems of independent interest.

LEMMA 1. <u>If</u> $c \prec c'$ <u>and</u> $c' \prec c$, <u>then</u> $c \sim c'$.

<u>Proof</u>: Let $\alpha_1, \cdots, \alpha_k$ be the distinct components of c, ordered so that $|\alpha_1| \geq \cdots \geq |\alpha_k|$. Let the multiplicity of α_ℓ be m_ℓ ($\Sigma\, m_\ell = n$), and assume that c has been arranged to take the form

$$(2.3) \qquad c = (\alpha_1, \cdots, \alpha_1, \cdots, \alpha_k, \cdots, \alpha_k)\,.$$

In view of the remark following Definition 1, the relations $c \prec c'$, $c' \prec c$ are still valid, hence there exist doubly stochastic matrices S, S' such that

$$(2.4) \qquad c = Sc' \quad \text{and} \quad c' = S'c\ ;$$

thus $c = SS'c$. Since the class of $n \times n$ doubly stochastic matrices form a multiplicative semigroup we have

$$c = Tc, \quad (T = SS') , \tag{2.5}$$

where T is doubly stochastic as well. We assert that

$$T = T_1 \oplus \cdots \oplus T_k , \tag{2.6}$$

where T_ℓ is doubly stochastic of order $m_\ell \times m_\ell$.

To prove (2.6) assume for simplicity that $k = 2$, i.e.,

$$c = (\alpha_1,\cdots,\alpha_1,\ \alpha_2,\cdots,\alpha_2), \quad \alpha_1 \neq \alpha_2, \quad |\alpha_1| \geq |\alpha_2| ,$$

where the multiplicity of α_ℓ $(\ell = 1,2)$, is m_ℓ and $m_1 + m_2 = n$. Take any of the first m_1 components of the equality in (2.5), say the i-th one. Since $|\alpha_1| \geq |\alpha_2|$ this leads to

$$|\alpha_1| = \left|\left(\sum_{j=1}^{m_1} T_{ij}\right)\alpha_1 + \left(\sum_{j=m_1+1}^{n} T_{ij}\right)\alpha_2\right|$$

$$\leq \left(\sum_{j=1}^{m_1} T_{ij}\right)|\alpha_1| + \left(\sum_{j=m_1+1}^{n} T_{ij}\right)|\alpha_2| \leq \left(\sum_{j=1}^{n} T_{ij}\right)|\alpha_1| = |\alpha_1| ,$$

Hence we have equality which, in view of the fact that $\alpha_1 \neq \alpha_2$, may hold if and only if $T_{ij} = 0$ for $j = m_1 + 1,\cdots,n$. This means that the first m_1 rows of T vanish beyond their m_1 entries, so all the weight of these rows is concentrated in the first m_1 columns. Consequently, the first m_1 columns of T vanish beyond their m_1 elements as well, and we obtain the desired decomposition $T = T_1 \oplus T_2$.

Next recall that doubly stochastic matrices are convex combinations of permutation matrices P_σ. In particular $S = \Sigma_\sigma \alpha_\sigma P_\sigma$, thus

$$T = SS' = \sum_\sigma \alpha_\sigma P_\sigma S' .$$

The matrices $\alpha_\sigma P_\sigma S'$ in the above sum have non-negative entries; hence they must all have the same block decomposition as T. Now we choose a coefficient α_τ with $\alpha_\tau \neq 0$, and conclude that $P_\tau S'$ decomposes according to (2.6). Since $P_\tau S'$ is doubly stochastic and it has the same decomposition (2.3) as c, it follows that $P_\tau S'c = c$. So, finally, by (2.4),

$$c' = S'c = (P_\tau^{-1})(P_\tau S'c) = P_\tau^{-1}c \sim c ,$$

and the lemma follows.

REMARK. The above proof contains a special case of the following observation on group-rings over the reals (or any ordered field). Let

$$R(G) = \{\Sigma \alpha_i g_i : \alpha_i \in \mathbb{R},\ g_i \in G\}$$

be a group-ring of G over $\mathbb{R}$, and let K_G be the convex hull of G in R(G), that is

$$K_G = \{ \Sigma\, \alpha_i g_i : \alpha_i \geq 0,\ \ \Sigma\, \alpha_i = 1 \} .$$

Then K_G is a multiplicative semigroup whose units are the elements of G. If H is a subgroup of G, then K_H is a sub-semigroup of K_G and two elements u,v of K_G satisfy $uv \in K_H$ if and only if there exists an element $g \in G$ such that ug and $g^{-1}v$ are in K_H. Thus the only divisors, in K_G, of elements of K_H are associates of elements of K_H.

We conclude this section with the following property of $<$ and $\ll$.

THEOREM 2. *The relations* $<$ *and* $\ll$ *are partial order relations on the set of unordered* n-*tuples*.

Proof. We have to show that $<$ and $\ll$ are reflexive, transitive and antisymmetric. The first two properties are easily verified, and by Theorem 1, $c \ll c'$ implies $c < c'$. So, it suffices to prove the antisymmetry of $<$, i.e., that $c < c'$ together with $c' < c$ yields $c \sim c'$. But this is the statement of Lemma 1, and the proof is complete.

3. Elementary inclusion relations

Before considering a general $n \times n$ case we present the following result concerning 2×2 matrices.

LEMMA 2. *If A is a 2×2 matrix, then for any α_1, α_2,*

$$(3.1)\qquad W_{(\alpha_1,\alpha_2)}(A) = (\alpha_1 - \alpha_2)\, W(A - \tfrac{1}{2}(\mathrm{tr}\, A)I) + \tfrac{1}{2}(\alpha_1 + \alpha_2)\{\mathrm{tr}\, A\} .$$

Thus $W_{(\alpha_1,\alpha_2)}(A)$ is convex.

Proof. As before let Λ_2 denote the set of all orthonormal pairs of 2-vectors. If x_1, x_2 is in Λ_2, then

$$(3.2)\qquad \alpha_1(A x_1,x_1) + \alpha_2(A x_2,x_2)$$

$$= \tfrac{1}{2}(\alpha_1 - \alpha_2)(A x_1,x_1) - \tfrac{1}{2}(\alpha_1 - \alpha_2)(A x_2,x_2) + \tfrac{1}{2}(\alpha_1 + \alpha_2)((A x_1,x_1) + (A x_2,x_2))$$

$$= (\alpha_1 - \alpha_2)(A x_1,x_1) - \tfrac{1}{2}(\alpha_1 - \alpha_2)\mathrm{tr}A + \tfrac{1}{2}(\alpha_1 + \alpha_2)\mathrm{tr}\, A$$

$$= (\alpha_1 - \alpha_2)((A - \tfrac{1}{2}(\mathrm{tr}\, A)I)x_1,x_1) + \tfrac{1}{2}(\alpha_1 + \alpha_2)\mathrm{tr}\, A .$$

So, (3.1) is obtained from (3.2) as x_1, x_2 vary over Λ_2.

The convexity of $W_{(\alpha_1,\alpha_2)}(A)$ is implied by the convexity of the (classical) numerical range and the lemma follows.

Using the above lemma we obtain our first general inclusion relation.

COROLLARY 1. <u>If</u> (γ_1,γ_2) <u>is obtained from</u> (γ_1',γ_2') <u>by pinching, then</u>

$$W_{(\gamma_1,\gamma_2)}(A) \subset W_{(\gamma_1',\gamma_2')}(A), \quad \forall A \in C_{2\times 2}. \tag{3.3}$$

<u>Proof</u>. By definition of pinching there exists an α, $0 \le \alpha \le 1$, such that

$$\gamma_1 = \alpha\gamma_1' + (1-\alpha)\gamma_2', \quad \gamma_2 = (1-\alpha)\gamma_1' + \alpha\gamma_2'.$$

Hence, by Lemma 2, the two sets in (3.3) are

$$W_{(\gamma_1',\gamma_2')}(A) = (\gamma_1' - \gamma_2')\, W(B) + \tfrac{1}{2}(\gamma_1' + \gamma_2')\{\text{tr } A\} \tag{3.4a}$$

and

$$W_{(\gamma_1,\gamma_2)}(A) = (2\alpha - 1)(\gamma_1' - \gamma_2')\, W(B) + \tfrac{1}{2}(\gamma_1' + \gamma_2')\{\text{tr } A\}, \tag{3.4b}$$

where $B = A - \frac{1}{2}(\text{tr } A)I$.

It is known (e.g., [1], §166) that the numerical range of any 2×2 matrix is an ellipse (possibly degenerate) with the eigenvalues as foci. That is, $W(B)$ is an ellipse centered at $(1/2)\text{tr } B$. In our case $\text{tr } B = 0$, so $(\gamma_1' - \gamma_2')W(B)$ is convex and symmetric with respect to the origin. Therefore, since $-1 \le 2\alpha - 1 \le 1$, we have

$$(2\alpha - 1)(\gamma_1' - \gamma_2')\, W(B) \subset (\gamma_1' - \gamma_2')\, W(B).$$

Hence the set in (3.4a) includes the set in (3.4b), and (3.3) follows.

LEMMA 3. <u>If</u> c <u>is obtained from</u> c' <u>by pinching, then,</u>

$$W_c(A) \subset W_{c'}(A), \quad \forall A \in C_{n\times n}. \tag{3.5}$$

Proof: Let i, j, $i < j$, be the pinching indices described in (2.1). Every fixed choice of $n - 2$ orthonormal vectors in $\mathbb{C}^n$,

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, \tag{3.6}$$

determines a 2-space, $\underset{\sim}{X}$, perpendicular to these vectors. The values of $W_c(A)$ and $W_{c'}(A)$ corresponding to the vectors in (3.6) are, respectively,

$$\sum_{\substack{k=1 \\ k \neq i,j}}^{n} \gamma_k (A x_k, x_k) + W_{(\gamma_i, \gamma_j)}(PA), \tag{3.7a}$$

and

$$\sum_{\substack{k=1 \\ k \neq i,j}}^{n} \gamma_k (A x_k, x_k) + W_{(\gamma_i', \gamma_j')}(PA). \tag{3.7b}$$

Here P is the projection of $\mathbb{C}^n$ on $\underset{\sim}{X}$, and it is understood that $W_{(\alpha,\beta)}(PA)$ is defined over $\underset{\sim}{X}$, i.e.,

$$W_{(\alpha,\beta)}(PA) = \{\alpha (A x, x) + \beta (A y, y) : x, y \in \underset{\sim}{X};\ \ x, y \ \text{ orthonormal}\}.$$

Since $\underset{\sim}{X}$ is 2-dimensional and PA maps $\underset{\sim}{X}$ into itself, the restriction of PA to $\underset{\sim}{X}$ may be presented by a 2×2 matrix. Moreover, it is clear from (2.1) that since c is a pinch of c', then (γ_i, γ_j) is obtained from (γ_i', γ_j') by the same pinching. Thus, Corollary 1 implies that

$$W_{(\gamma_i, \gamma_j)}(PA) \subset W_{(\gamma_i', \gamma_j')}(PA).$$

Consequently, the set in (3.7b) includes the set of (3.7a). Since the vectors in (3.6) were arbitrary, relation (3.5) holds and the proof is complete.

The following theorem is an immediate consequence of Lemma 3.

THEOREM 3. <u>If</u> $c \ll c'$, <u>then</u>

$$W_c(A) \subset W_{c'}(A), \qquad \forall A \in \mathbb{C}_{n\times n} . \tag{3.8}$$

<u>Proof</u>. By hypothesis, there exists a finite sequence, $c' = c_1, c_2, \cdots, c_\ell = c$, such that each c_i $(1 < i \leq \ell)$, is obtained from c_{i-1} by pinching. So, by Lemma 3,

$$W_c(A) = W_{c_\ell}(A) \subset \cdots \subset W_{c_1}(A) = W_{c'}(A), \qquad \forall A \in \mathbb{C}_{n\times n} ,$$

and (3.8) follows.

At this point it would be natural to ask whether $c \prec c'$ implies (3.8) or not. To answer this question in the negative take $A = \mathrm{diag}\,(0,1,i)$ and $c' = (0,1,i)$. Westwick [2], has shown that $W_{c'}(A)$ includes the points 1 and $2i$, but not the open line segment joining them. In particular $(1 + 2i)/2 \notin W_{c'}$. Now take $c = (1/2, i/2, 1/2 + i/2)$. By (2.2) we have that $c \prec c'$; yet the point

$$\gamma_1(A\ e_3, e_3) + \gamma_2(A\ e_1, e_1) + \gamma_3(A\ e_2, e_2) = \frac{1 + 2i}{2}$$

of $W_c(A)$ does not belong to $W_{c'}(A)$.

A somewhat weaker result holds for the relation $\prec$, and we establish first the next lemma.

LEMMA 4. <u>Given two bounded disjoint convex sets</u> K_1, K_2 <u>in</u> $\mathbb{C}^n$, <u>then there exists a linear functional</u> φ <u>on</u> $\mathbb{C}^n$, <u>such that</u> $\varphi(x) \neq \varphi(y)$ <u>for all</u> $x \in K_1$, $y \in K_2$.

<u>Proof</u>. We first consider K_1, K_2 as convex sets in $\mathbb{R}^{2n}$. By the Separation

Theorem for real vector spaces (e.g., [5], Theorem 20, p. 204), there exists a linear real functional $\psi(x)$ on $\mathbb{R}^{2n}$, such that $\psi(x) < \psi(y)$ for all $x \in \mathcal{K}_1$, $y \in \mathcal{K}_2$. More explicitly we have

$$\psi(x) = \beta_{11}\xi_{11} + \beta_{12}\xi_{12} + \beta_{21}\xi_{21} + \beta_{22}\xi_{22} + \cdots + \beta_{n1}\xi_{n1} + \beta_{n2}\xi_{n2}$$

where $x = (\xi_1,\cdots,\xi_n)$, $\xi_j = \xi_{j1} + i\xi_{j2}$, and the β_{ij} are real coefficients. Now define a complex functional on $\mathbf{C}^n$:

$$\varphi(x) = \beta_1\xi_1 + \cdots + \beta_n\xi_n, \qquad \beta_j = \beta_{j1} - i\beta_{j2} .$$

It is easily seen that $\psi(x) = \mathrm{Re}(\varphi(x))$; so $\mathrm{Re}(\varphi(x)) < \mathrm{Re}(\varphi(y))$ for $x \in \mathcal{K}_1$, $y \in \mathcal{K}_2$, and the lemma follows.

THEOREM 4. <u>We have</u> $c \prec c'$ <u>if and only if</u>

$$(3.9) \qquad W_c(A) \subset \mathrm{conv}\{W_{c'}(A)\}, \qquad \forall A \in \mathbf{C}_{n\times n} .$$

<u>Proof</u>. If $c \prec c'$, then, for some doubly stochastic S, we have $c = Sc'$. The matrix S is a convex combination of permutation matrices P_σ. Thus $c = \Sigma_\sigma \alpha_\sigma P_\sigma c'$, and the relation among the components of c and c' is

$$\gamma_j = \sum_\sigma \alpha_\sigma \gamma'_{\sigma(j)}, \qquad j = 1,\cdots,n .$$

This yields that any point $\Sigma\, \gamma_j (A x_j, x_j)$ of $W_c(A)$ satisfies

$$\Sigma\, \gamma_j (A x_j, x_j) = \sum_{j=1}^{n} (A x_j, x_j) \sum_\sigma \alpha_\sigma \gamma'_{\sigma(j)}$$

$$= \sum_\sigma \alpha_\sigma \left[\sum_{j=1}^{n} \gamma_{\sigma(j)} (A x_j, x_j) \right] .$$

That is, each point in W_c is a convex combination of points in $W_{c'}$ and (3.9) follows.

For the necessity part of the proof we recall that the condition $c < c'$ is equivalent to the fact that c belongs to the convex set

$$\mathcal{K}_1 = \{Sc' : S = \text{doubly stochastic}\} .$$

Let $\mathcal{K}_2$ be the set which consists only of c. If $c \not< c'$, then $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$, and by Lemma 4 there exist complex coefficients $\beta_1, \cdots, \beta_n$ such that the linear functional $\varphi(x) = \Sigma_i \beta_i \xi_i$ satisfies

$$(3.10) \quad \varphi(c) \notin \{\varphi(x) : x \in \mathcal{K}_1\} = \{\varphi(Sc') : S = \text{doubly stochastic}\} .$$

Consider now the matrix $B = \mathrm{diag}(\beta_1, \cdots, \beta_n)$. We have

$$(3.11) \qquad \varphi(c) = \sum_{j=1}^{n} \beta_j \gamma_j = \Sigma\, \gamma_j (B e_j, e_j) \in W_c(B) .$$

On the other hand take any point $\Sigma_j \gamma_j' (B x_j, x_j)$ in $W_{c'}(B)$. Here $\{x_j = (\xi_{1j}, \cdots, \xi_{nj})\}_{j=1}^{n}$ is an orthonormal system in $\mathbf{C}^n$, so $\Sigma_i |\xi_{ij}|^2 = \Sigma_j |\xi_{ij}|^2 = 1$ and consequently the matrix X, with $X_{ij} = |\xi_{ij}|^2$, is doubly stochastic. Hence

$$\sum_{j=1}^{n} \gamma_j' (B x_j, x_j) = \sum_{j=1}^{n} \gamma_j' \sum_{i=1}^{n} \beta_i |\xi_{ij}|^2$$

$$= \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} X_{ij} \gamma_j' = \varphi(Xc') .$$

This gives

$$W_{c'}(B) \subset \{\varphi(Sc') : S = \text{doubly stochastic}\} ,$$

and since the set on the right side is convex, we get in fact

$$(3.12) \qquad \mathrm{conv}\, W_{c'}(B) \subset \{\varphi(Sc') : S = \text{doubly stochastic}\} .$$

The inclusion in (3.12) together with (3.10), (3.11) yields $W_c(B) \not\subset \text{conv } W_{c'}(B)$, and (3.9) is violated.

4. The case of real coefficients

For real vectors c the situation is much simpler. As in the complex case, the set $W_c(A)$ remains unchanged under permutations of the γ_j. Therefore, given a set of coefficients $\{\gamma_1,\cdots,\gamma_n\}$, it will often be convenient to arrange them in decreasing order.

DEFINITION 2. A real vector $c = (\gamma_1,\cdots,\gamma_n)$ is called <u>ordered</u> if

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \,.$$

The convenience of ordering real vectors is demonstrated in the next lemma.

LEMMA 5. *If* c' *is ordered and* $c < c'$, *then*

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma_j', \qquad k = 1,\cdots,n \,, \tag{4.1}$$

with equality for $k = n$.

Proof. If $c < c'$ then for some doubly stochastic S, we have $c = Sc'$. Hence for a fixed k, $1 \leq k \leq n$,

$$\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \sum_{j=1}^{n} s_{ij}\gamma_j' = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} s_{ij} \right) \gamma_j' \,. \tag{4.2}$$

Setting

$$\alpha_j = \sum_{i=1}^{k} s_{ij}, \qquad j = 1,2,\cdots,n \,,$$

we have

$$(4.3) \qquad 0 \leq \alpha_j \leq 1 \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j = k .$$

So, using the fact that c' is ordered, we get from (4.2), (4.3)

$$\sum_{i=1}^{k} \gamma_i = \sum_{j=1}^{n} \alpha_j \gamma'_j \leq \sum_{j=1}^{k} \gamma'_j .$$

For $k = n$ each $\alpha_j = 1$ and we have equality.

We remark that the relations in (4.1) are discussed in Chapter 2 of [6], beginning with Section 2.18.

Two more preliminary result leads to Theorem 5.

LEMMA 6. _Let_ γ'_i, γ'_j _with_ $\gamma'_i > \gamma'_j$ _be two real components of_ c'. _Let_ δ _satisfy_ $0 \leq \delta \leq \gamma'_i - \gamma'_j$. _Then_

$$c \equiv c' - \delta(e_i - e_j)$$

is a pinch of c'.

Proof. Denote $\alpha' = \delta/(\gamma'_i - \gamma'_j)$. Evidently $0 \leq \alpha' \leq 1$, and by the definition of c we have

$$(4.4a) \qquad \gamma_i = \gamma'_i - \delta = \gamma'_i - \alpha'(\gamma'_i - \gamma'_j) = (1 - \alpha')\gamma'_i + \alpha'\gamma'_j$$

and

$$(4.4b) \qquad \gamma_j = \gamma'_j + \delta = \gamma'_j + \alpha'(\gamma'_i - \gamma'_j) = \alpha'\gamma'_i + (1 - \alpha')\gamma'_j .$$

Equations (4.4) are equivalent to (2.1), hence c is a pinch of c' and the statement is proven.

LEMMA 7. Let c, c' be ordered. If c, c' satisfy (4.1) with equality for $k = n$, then $c \ll c'$.

Proof. The idea of the proof is to construct a sequence of vectors $c' = c_1, c_2, \cdots$, such that each c_i has the following three properties. First,

$$c_i \ll c_{i-1}, \qquad i \geq 2 ; \tag{4.5}$$

second,

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma_{ij}, \qquad k = 1,2,\cdots,n , \tag{4.6}$$

with equality for $k = n$; and third, the number of equal elements in the sets $\{\gamma_{i1}, \cdots \gamma_{in}\}$ and $\{\gamma_1, \cdots, \gamma_n\}$ is at least $i - 1$. Here the γ_j and γ_{ij} are, respectively, the components of c and c_i.

By the last property, there exists a finite ℓ $(\ell \leq n)$, for which $c_\ell = c$. Hence, by property (4.5) we get

$$c = c_\ell \ll \cdots \ll c_1 = c', \tag{4.7}$$

which leads by transitivity to the desired result $c \ll c'$.

As indicated, we start by choosing $c_1 = c'$, for which the first and third properties are satisfied in a trivial manner. To show the second, we use the hypothesis $c < c'$ with Lemma 5, and find that c and $c_1 \equiv c'$ satisfy (4.6).

Now suppose that $c_1, \cdots, c_i$ with the above properties has been constructed. If $c_i = c$, then the sequence (4.7) is complete; so let us assume $c_i \neq c$ and construct c_{i+1}. We have the inequalities in (4.6) from which we conclude that there exist an r, $1 \leq r < n$, so that

(4.8a) $$\gamma_1 = \gamma_{i1}, \cdots, \gamma_{r-1} = \gamma_{i,r-1} \; ; \; \gamma_r < \gamma_{ir} ,$$

and a least s, $r < s \leq n$, such that

(4.8b) $$\gamma_s > \gamma_{is} .$$

Since c is ordered, we have $\gamma_r \geq \gamma_s$, which together with (4.8) gives $\gamma_{ir} > \gamma_r \geq \gamma_s > \gamma_{is}$. So the quantity

(4.9) $$\delta = \min\{\gamma_{ir} - \gamma_r, \; \gamma_s - \gamma_{is}\}$$

satisfies $0 < \delta < \gamma_{ir} - \gamma_{is}$. Hence, by Lemma 6,

(4.10) $$c_{i+1} \equiv c_i - \delta(e_r - e_s)$$

is a pinch of c_i. So $c_{i+1} \ll c_i$, i.e., c_{i+1} has the first property (4.5).

Next, we wish to show that c_{i+1} has the second property, that is

$$\sum_{j=1}^{k} \gamma_i \leq \sum_{j=1}^{k} \gamma_{i+1,j} \qquad k = 1, \cdots, n ,$$

with equality for $k = n$. Since c_i satisfies (4.6), and since c_{i+1} is obtained from c_i by changing only the r and s components while their sum is preserved, it is clear that for any k with $1 \leq k < r$ or $s \leq k \leq n$, we have

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma_{ij} = \sum_{j=1}^{k} \gamma_{i+1,j} .$$

Now use (4.8) - (4.10) to find that

$$\gamma_{i+1,r} = \gamma_{ir} - \delta \geq \gamma_{ir} - (\gamma_{ir} - \gamma_r) = \gamma_r .$$

So, also for $r \leq k < s$,

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma_{ij} \leq \sum_{j=1}^{k} \gamma_{i+1,j} .$$

Finally, consider the third property. According to the construction of c_{i+1}, we have $\gamma_{i+1,r} = \gamma_r$ or $\gamma_{i+1,s} = \gamma_s$, or both. So, by comparing with (4.8) we see that the number of components of c_{i+1} which equal components of c is greater than the number of equalities for c_i and c, and is therefore at least i. This completes the proof.

Combining Lemmas 5 and 7, together with Theorem 1, we easily obtain the following.

THEOREM 5. _Let_ c, c' _be ordered vectors. Then each of the relations_ $c < c'$ _and_ $c \ll c'$ _is equivalent to_

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma'_j , \qquad k = 1,\cdots,n , \tag{4.11}$$

with equality for $k = n$.

In general, it is more convenient to verify condition (4.11), than to check whether $c < c'$ or $c \ll c'$ according to the original definitions.

Since the relations $c < c'$ and $c \ll c'$ are preserved under permutations of the γ_j, γ'_j, we rephrase part of Theorem 5:

THEOREM 6. _If_ c, c' _are real vectors, then the relations_ $c < c'$ _and_ $c \ll c'$ _are equivalent._

We come now to one of the main results.

THEOREM 7. If c, c' are real, then $c \prec c'$ if and only if

$$W_c(A) \subset W_{c'}(A), \quad \forall A \in C_{n\times n}. \tag{4.12}$$

Proof. By Theorem 6, $c \prec c'$ implies $c \ll c'$, so by Theorem 3 we have (4.12). Conversely, (4.12) yields (3.9) and by Theorem 4, $c \prec c'$.

REMARK. Theorem 7 can be obtained immediately from Theorem 4, using the fact that for real c, W_c is convex, i.e., $W_c = \mathrm{conv}\{W_c\}$. Yet, the convexity if W_c is not essential to the proof.

COROLLARY 2. (a) If $c = (\gamma_1, \cdots, \gamma_n) \in \mathbb{R}^n$ with $\Sigma_j \gamma_j = \alpha$, then $(\alpha/n, \cdots, \alpha/n) \prec c$ and hence

$$\left\{\frac{\alpha}{n} \operatorname{tr} A\right\} \subset W_c(A), \quad \forall A \in C_{n\times n}. \tag{4.13}$$

(b) If $\gamma_j \geq 0$, then $c \prec (\alpha, 0, \cdots, 0)$ and

$$W_c(A) \subset \alpha\, W(A), \quad \forall A \in C_{n\times n}.$$

(c) If $\alpha = 0$ then

$$\bigcap_{A \in C_{n\times n}} W_c(A) = \{0\}. \tag{4.14}$$

Proof. First take the ordered version of c and observe that Theorems 5, 7 yield (a) and (b). Now, if $\alpha = 0$, then according to (4.13), $0 \in W_c(A)$ for all A. Since $W_c(0) = \{0\}$ we have (4.14) and the corollary follows.

COROLLARY 3. <u>(Fillmore and Williams). The</u> k-<u>numerical ranges satisfy</u>

$$\left\{\frac{1}{n}\operatorname{tr} A\right\} = W_n(A) \subset \cdots \subset W_2(A) \subset W_1(A) \equiv W(A) \ . \tag{4.15}$$

<u>Proof</u>: By (1.1), $W_s(A) = W_{c_s}(A)$ with the ordered vector

$$c_s = (\gamma_{s1}, \cdots, \gamma_{sn}) = \frac{1}{s}(e_1 + \cdots + e_s) \ .$$

For all $1 \leq s \leq n$ we have

$$\sum_{j=1}^{k} \gamma_{sj} = \frac{1}{s}\min\{k,s\} \geq \frac{1}{s+1}\min\{k, s+1\} = \sum_{j=1}^{k} \gamma_{s+1,j} \ ,$$

with equality for $k = n$. So Theorem 5 implies that $c_{s+1} < c_s$, $1 \leq s \leq n$. Hence, by Theorem 7,

$$W_{s+1}(A) = W_{c_{s+1}}(A) \subset W_{c_s}(A) = W_s(A) \ ; \qquad s = 1, \cdots, n-1 \ ,$$

and we get (4.15).

This result was obtained in a different way, using the convexity of W_k, by Fillmore and Williams, [7].

REMARK. In general, for given vectors $c = (\gamma_1, \cdots, \gamma_n)$, $c' = (\gamma_1', \cdots, \gamma_n')$, there exists no constant λ such that $c < \lambda c'$. To demonstrate this statement assume that c, c' are ordered and that $\Sigma\, \gamma_j' > 0$, $\Sigma\, \gamma_j \geq 0$. If $c < \lambda c'$, then for some doubly stochastic S we would have $c = \lambda S c'$, which yields

$$\sum_{i=1}^{n} \gamma_i = \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}\gamma_j' = \lambda \sum_{j=1}^{n} \gamma_j' \ .$$

Consequently

$$\lambda = \Sigma \gamma_i \,/\, \Sigma \gamma_j' , \tag{4.16}$$

so $\lambda \geq 0$, and $\lambda c'$ is ordered. Now, by Theorem 5 we should get

$$\sum_{j=1}^{k} \gamma_j \leq \lambda \sum_{j=1}^{k} \gamma_j' , \qquad k = 1,\cdots,n , \tag{4.17}$$

with equality for $k = n$. But as λ of (4.16) satisfies (4.17) for $k = n$, it will not, in general, satisfy the rest of (4.17).

The situation is quite different in the homogeneous case $\Sigma \gamma_j = \Sigma \gamma_j' = 0$, where we have the following result.

LEMMA 8. <u>Let</u> c, c' <u>be ordered vectors with</u> $\Sigma_j \gamma_j = \Sigma_j \gamma_j' = 0$ <u>and</u> $c' \neq 0$. Set

$$\eta = \eta(c,c') = \max_{1 \leq k < n} \frac{\gamma_1 + \cdots + \gamma_k}{\gamma_1' + \cdots + \gamma_k'} , \tag{4.18a}$$

$$\zeta = \zeta(c,c') = \min_{1 \leq k < n} \frac{\gamma_1 + \cdots + \gamma_k}{\gamma_n' + \cdots + \gamma_{n-k+1}'} . \tag{4.18b}$$

<u>Then</u> $c < \lambda c'$ <u>if and only if</u> $\lambda \geq \eta$ <u>or</u> $\lambda \leq \zeta$.

<u>Proof</u>. First we show that

$$\gamma_1' + \cdots + \gamma_k' > 0, \quad \gamma_n' + \cdots + \gamma_{n-k+1}' < 0 ; \qquad k = 1,\cdots,n-1 .$$

Since $\Sigma \gamma_j' = 0$, it suffices to prove the left inequalities, so assume that $\gamma_1' + \cdots + \gamma_k' \leq 0$ for some $k < n$. This means that $\gamma_{k+1}' + \cdots + \gamma_n' \geq 0$, thus $\gamma_{k+1}' \geq 0$, and consequently $\gamma_1' \geq \cdots \geq \gamma_k' \geq \gamma_{k+1}' \geq 0$. Since $c' \neq 0$, we have $\gamma_1' > 0$ and our assumption is contradicted. Similarly, the partial sums $\gamma_1 + \cdots + \gamma_k$, $k < n$, are non-negative, and it follows that η, ζ of (4.18) are well defined and satisfy $\eta \geq 0$, $\zeta \leq 0$.

Now choose λ with $\lambda \geq 0$. The vector $\lambda c'$ remains ordered, and according to Theorem 5, $c \prec \lambda c'$ if and only if

$$\lambda \sum_{j=1}^{k} \gamma_j' \geq \sum_{j=1}^{k} \gamma_j \ ; \qquad k = 1,\cdots,n \ ,$$

with equality for $k = n$. The hypothesis $\Sigma\, \gamma_j = \Sigma\, \gamma_j' = 0$ implies equality for $k = n$; so $c \prec \lambda c'$ is equivalent to

$$\lambda \sum_{j=1}^{k} \gamma_j' \geq \sum_{j=1}^{k} \gamma_j, \qquad k = 1,\cdots,n-1 \ . \tag{4.19}$$

However, by the definition of η,

$$\eta \sum_{j=1}^{k} \gamma_j' \geq \sum_{j=1}^{k} \gamma_j \ , \qquad k = 1,\cdots,n-1 \ ,$$

with equality for some $1 \leq k < n$. Thus, (4.19) holds if and only if $\lambda \geq \eta$.

If $\lambda < 0$, then $\lambda c'$ becomes unordered, and its equivalent ordered version with a positive multiplier is $(-\lambda)(-\gamma_n',\cdots,-\gamma_1')$. Using the previous argument, we find that $c \prec \lambda c'$ if and only if

$$-\lambda \geq \max_{1 \leq k < n} \frac{\gamma_1 + \cdots + \gamma_k}{-\gamma_n' - \cdots - \gamma_{n-k+1}'} = -\min_{1 \leq k < n} \frac{\gamma_1 + \cdots + \gamma_k}{\gamma_n' + \cdots + \gamma_{n-k+1}'} = -\zeta \ ,$$

and the lemma follows.

Theorem 7 and Lemma 8 have an immediate consequence.

THEOREM 8. _Let_ c, c' _be ordered vectors with_ $\Sigma_j\, \gamma_j = \Sigma_j\, \gamma_j' = 0$ _and_

$c' \neq 0$. *Then*

$$W_c(A) \subset W_{\lambda c'}(A) \equiv \lambda W_{c'}(A)\ , \qquad \forall A \in C_{n\times n}\ ,$$

if and only if $\lambda \geq \eta(c,c')$ *or* $\lambda \leq \zeta(c,c')$ *where* η, ζ *are defined in* (4.18).

COROLLARY 4. *Let* $a = (\alpha_1,\cdots,\alpha_n)$ *and* $a' = (\alpha_1',\cdots,\alpha_n')$ *be ordered vectors such that not all the components of* a' *are equal. Set* $\alpha = \Sigma\,\alpha_j$, $\alpha' = \Sigma\,\alpha_j'$, and define

$$c = a - (\alpha/n,\cdots,\alpha/n)\ , \quad c' = a' - (\alpha'/n,\cdots,\alpha'/n)\ .$$

Then,

$$W_a(A) - \left\{\frac{\alpha}{n}\ \mathrm{tr}\ A\right\} \subset \lambda\left(W_{a'}(A) - \left\{\frac{\alpha'}{n}\ \mathrm{tr}\ A\right\}\right)\ , \qquad \forall A \in C_{n\times n}\ ,$$

if and only if $\lambda \geq \eta(c,c')$ *or* $\lambda \leq \zeta(c,c')$, *where* η, ζ *are given in* (4.18).

Proof. The components of the vectors c, c' satisfy $\Sigma\,\gamma_j = \Sigma\,\gamma_j' = 0$, and $c' \neq 0$. Hence, by Theorem 8.

$$W_a(A) - \left\{\frac{\alpha}{n}\ \mathrm{tr}\ A\right\} = W_c(A) \subset \lambda W_{c'}(A) = \lambda\left(W_{a'}(A) - \frac{\alpha'}{n}\ \mathrm{tr}\ A\right)\ , \qquad \forall A \in C_{n\times n}\ ,$$

if and only if the conditions of the corollary are satisfied.

5. Generalized numerical radius

A concept which directly relates to the generalized numerical range

$W_c(A)$, is the generalized numerical radius

$$r_c(A) = \max \{|z| : z \in W_c(A)\}$$

$$= \max \left\{ \left| \sum_{j=1}^{n} \gamma_j (A x_j, x_j) \right| : (x_1, \cdots, x_n) \in \Lambda_n \right\}.$$

In particular we have the k-numerical radius

$$r_k(A) = \max \{|z| : z \in W_k(A)\}, \qquad k = 1, 2, \cdots, n,$$

which reduces, for $k = 1$, to the classical numerical radius

$$r(A) = \max \{|z| : z \in W(A)\} = \max_{|x|=1} |(A x, x)|.$$

The function $r(A)$ provides an important tool in the linear stability analysis of multidimensional hyperbolic and parabolic initial value problems (e.g., [8] §2), and one may expect that the generalized radius will be applicable as well.

It is obvious that if $W_c(A) \subset W_{c'}(A)$ or even if $W_c(A) \subset \operatorname{conv} W_{c'}(A)$, then $r_c(A) \leq r_{c'}(A)$, though the converse may fail to hold. Thus, we use Theorems 4, 8 and Corollaries 2,3, to obtain, respectively, the following results.

THEOREM 9. (a) If c, c' are complex n-vectors with $c \prec c'$, then

(5.1) $$r_c(A) \leq r_{c'}(A), \qquad \forall A \in C_{n\times n}.$$

(b) Let c, c' be real ordered vectors with $\Sigma \gamma_j = \Sigma \gamma'_j = 0$ and $c' \neq 0$. Let λ satisfy $\lambda \geq \eta(c,c')$ or $\lambda \leq \zeta(c,c')$ where η, ζ are defined in (4.18). Then

$$r_c(A) \leq |\lambda| \, r_{c'}(A), \qquad \forall A \in C_{n\times n}.$$

(c) <u>For</u> $c = (\gamma_1, \cdots, \gamma_n)$ <u>real with</u> $\Sigma_j \gamma_j = \alpha$,

$$\frac{|\alpha|}{n} |\mathrm{tr}\, A| \leq r_c(A), \qquad \forall A \in \mathbf{C}_{n\times n}.$$

<u>If</u> $\gamma_j \geq 0$, <u>then</u>

$$r_c(A) \leq \alpha\, r(A), \qquad \forall A \in \mathbf{C}_{n\times n}.$$

(d) <u>The</u> k-<u>numerical radii satisfy</u>

$$\frac{1}{n} |\mathrm{tr}\, A| = r_n(A) \leq \cdots \leq r_1(A) = r(A), \qquad \forall A \in \mathbf{C}_{n\times n}.$$

6. C-numerical ranges

The numerical ranges defined in this paper can be generalized in the following way.

DEFINITION 3. Let $C \in \mathbf{C}_{n\times n}$ be fixed and let $\mathcal{U}_n$ denote the group of $n \times n$ unitary matrices. We call the set

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \in \mathcal{U}_n\}$$

the <u>C-numerical range</u> of the n-square matrix A.

If $c = (\gamma_1, \cdots, \gamma_n)$ is a given vector, we take $D = \mathrm{diag}\,(\gamma_1, \cdots, \gamma_n)$ and find that

$$(6.1) \qquad W_c(A) = \left\{ \sum_{j=1}^{n} \gamma_j x_j^* A x_j : (x_1, \cdots, x_n) \in \Lambda_n \right\}$$

$$= \{\mathrm{tr}\,(DU^*AU) : U \in \mathcal{U}_n\} = W_D(A).$$

So, indeed, $W_c(A)$ is a special case of the C-numerical range. In fact our last result will characterize the class of matrices C for which

$$W_C(A) = W_c(A) , \quad \forall A \in \mathbf{C}_{n\times n} .$$

First, we give two simple properties of the C-numerical range.

LEMMA 9. (a) For any $C, A \in C_{n\times n}$ we have

$$(6.2) \qquad W_C(A) = W_A(C) .$$

(b) The set $W_C(A)$ is invariant under unitary similarities of C or of A.

Proof. We have

$$W_C(A) = \{\mathrm{tr}\ (CU^*AU) : U \in \mathcal{U}_n\} = \{\mathrm{tr}\ (U^*AUC) : U \in \mathcal{U}_n\}$$

$$= \{\mathrm{tr}\ (AUCU^*) : U \in \mathcal{U}_n\} = W_A(C) ,$$

so (6.2) holds, and it follows that C and A play a symmetric role in the definition of $W_C(A)$. Hence, for part (b), it suffices to show that $W_C(A)$ is invariant under unitary similarities of A. But that is an immediate consequence of Definition 3 which states that $W_C(A)$ depends only on the class, $\mathcal{S}(A) = \{U^*AU : U \in \mathcal{U}_n\}$, of matrices unitarily similar to A.

The next result leads to Theorem 10.

LEMMA 10. If $\mathcal{S}, \mathcal{S}'$ are compact connected subsets of $C_{n\times n}$ so that

$$(6.3) \qquad \{\varphi(X) : X \in \mathcal{S}\} = \{\varphi(X') : X' \in \mathcal{S}'\}$$

<u>for all linear functionals</u> φ <u>on</u> $C_{n\times n}$, then

$$\mathcal{K} \equiv \operatorname{conv}\{\mathcal{S}\} = \operatorname{conv}\{\mathcal{S}'\} \equiv \mathcal{K}' .$$

<u>Proof</u>. We recall that the hyperplanes (of real dimension $2n^2 - 1$) of $C_{n\times n}$ are the loci of the equations

$$\operatorname{Re}(\varphi(X)) = \alpha$$

as φ varies over the nonzero functionals in $C^*_{n\times n}$ and α varies in $\mathbb{R}$. Since $\mathcal{S}$ is connected, a hyperplane intersects $\mathcal{S}$ if and only if it intersects $\operatorname{conv}\{\mathcal{S}\}$; thus (6.3) implies

$$(6.4) \qquad \{\operatorname{Re}(\varphi(X)) : X \in \mathcal{K}\} = \{\operatorname{Re}(\varphi(X')) : X' \in \mathcal{K}'\} , \qquad \forall \varphi \in C^*_{n\times n} .$$

Now choose a functional φ and consider the set of real values

$$\mathcal{R}_\varphi(\mathcal{K}) = \{\operatorname{Re}(\varphi(X)) : X \in \mathcal{K}\} .$$

Since $\mathcal{K}$ is compact and connected, $\mathcal{R}_\varphi(\mathcal{K})$ is a closed interval with end points

$$\mu_1 = \min \mathcal{R}_\varphi(\mathcal{K}) , \quad \mu_2 = \max \mathcal{R}_\varphi(\mathcal{K}) .$$

This means that a hyperplane $\operatorname{Re}(\varphi(X)) = \alpha$ intersects $\mathcal{K}$ if and only if $\alpha \in \mathcal{R}_\varphi(\mathcal{K})$, and in particular

$$(6.5) \qquad \operatorname{Re}(\varphi(X)) = \mu_1 , \quad \operatorname{Re}(\varphi(X)) = \mu_2 ,$$

are the two planes of support for $\mathcal{K}$ defined by φ.

According to (6.4)

$$(6.6) \qquad \mathcal{R}_\varphi(\mathcal{K}) = \mathcal{R}_\varphi(\mathcal{K}') , \qquad \forall \varphi \in C^*_{n\times n} ;$$

so the hyperplanes in (6.5) support $\mathcal{K}'$ as well as $\mathcal{K}$, for all φ. Since convex sets are uniquely determined by their supporting planes, the proof is complete.

THEOREM 10. *We have*

$$W_C(A) = W_{C'}(A) , \quad \forall A \in C_{n\times n} , \tag{6.7}$$

if and only if C, C' *are unitarily similar*.

Proof. If C, C' are unitarily similar, then (6.7) is given by part (b) of Lemma 9.

For the converse we use (a) of Lemma 9 by which the hypothesis in (6.7) becomes $W_A(C) = W_A(C')$ for all A; or more explicitly

$$\{\mathrm{tr}\,(AU^*CU) : U \in \mathcal{U}_n\} = \{\mathrm{tr}\,(AU^*C'U) : U \in \mathcal{U}_n\} , \quad \forall A \in C_{n\times n} . \tag{6.8}$$

Next we remember that every linear functional φ on $C_{n\times n}$ is of the form

$$\varphi(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}\xi_{ji} = \mathrm{tr}\,(AX) ,$$

where $A = [\alpha_{ij}]$ is a matrix of coefficients, and $X = [\xi_{ij}]$ is arbitrary. Thus, the hypothesis in (6.8) takes the form

$$\{\varphi(X) : X \in \mathcal{S}\} = \{\varphi(X') : X' \in \mathcal{S}'\} , \quad \forall \varphi \in C^*_{n\times n} ;$$

where

$$\mathcal{S} = \{U^*CU : U \in \mathcal{U}_n\} , \quad \mathcal{S}' = \{U^*C'U : U \in \mathcal{U}_n\}$$

are compact connected subsets of $C_{n\times n}$. Consequently, by Lemma 10, (6.9) $$\mathcal{K} \equiv \mathrm{conv}\{\mathcal{S}\} = \mathrm{conv}\{\mathcal{S}'\} \equiv \mathcal{K}' .$$

The sets $\mathcal{K}$, $\mathcal{K}'$ are compact, so they are spanned by the extreme points of $\mathcal{S}$ and $\mathcal{S}'$, respectively. Therefore, by the equality in (6.9) we finally get

$$\mathrm{ext}\{\mathcal{S}\} = \mathrm{ext}\{\mathcal{S}'\} .$$

Now take a point $U_1^*CU_1$ in $\mathrm{ext}\{\mathcal{S}\}$. It equals a point $U_2^*C'U_2$ in $\mathrm{ext}\{\mathcal{S}'\}$ where U_1, U_2 are both unitary. That is,

$$C = U^*C'U \quad \text{with} \quad U = U_2U_1^* ,$$

and the theorem is proven.

Our last result characterizes the relation between the C-numerical and the c-numerical ranges.

COROLLARY 5. For a given $C \in \mathbb{C}_{n\times n}$, there exists a vector $c \in \mathbb{C}^n$ such that

(6.10) $$W_C(A) = W_c(A) , \quad \forall A \in \mathbb{C}_{n\times n} ,$$

if and only if C is normal. If C is normal, then the components of c are the eigenvalues of C in an arbitrary order.

Proof. By (6.1), the equality in (6.10) is equivalent to having a diagonal $D = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ such that

$$W_C(A) = W_D(A) , \quad \forall A \in \mathbb{C}_{n\times n} .$$

But C is unitarily similar to a diagonal matrix, if and only if C is normal, so Theorem 10 completes the proof.

Note that if C is normal with real eigenvalues — that is Hermitian — then (6.10) holds with a real c, and by Westwick's Theorem $W_C(A)$ is convex.

We conclude this paper with the following discussion.

REMARK. It is clear now that $W_C(A)$ is the range of values of the mapping

$$\varphi : \mathcal{S}(A) \to \mathbf{C}$$

where

$$\mathcal{S}(A) = \{U^*AU : U \in \mathcal{U}_n\} \subset \mathbf{C}_{n\times n} ,$$

and φ is the linear functional on $\mathbf{C}_{n\times n}$ defined by

$$\varphi(X) = \mathrm{tr}\ (CX) .$$

That is, $W_C(A)$ gives us <u>all</u> the information a single functional can provide about the set $\mathcal{S}(A)$. From this point of view, $W_C(A)$ is an ultimate generalization of previous concepts of numerical ranges.

However, more information on $\mathcal{S}(A)$ could be obtained by considering mappings of the form

$$X \to (\varphi_1(X),\cdots,\varphi_m(X)) \in \mathbf{C}^m , \qquad (X \in \mathcal{S}(A)) ,$$

where $\varphi_1,\cdots,\varphi_m$ are functionals on $\mathbf{C}_{n\times n}$, and m is arbitrary. In fact we do not need $m > n^2$; for if we denote by φ_{ij} the functional defined by

$$\varphi_{ij}(X) = X_{ij} \equiv \xi_{ij} ,$$

then the mapping

$$X \to (\varphi_{11}(X),\cdots,\varphi_{nn}(X)) = (\xi_{11},\cdots,\xi_{nn}) \in C^{n^2}$$

exactly characterizes the set $\mathcal{S}(A)$.

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